S&K CLASSES

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TARGET XII CBSE BOARD 2018

1 Mark Questions

1. Calculate the direction cosines of the vector $\vec{a} = 3i - 2j + 5k$.
2. Find the value of $\cos\left\{cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\Pi}{6}\right\}$ ***[Ans. 1]***
3. For what value of x, $f(x) = |2x - 7|$ is not derivable. ***[Ans. $x = -7/2$]***
4. Find $|adj\ (A)|$, If $|A|$ = 7 of 3 x 3 matrix.
5. Write the set of points of continuity of $g(x) = |x - 1| + |x + 1|$.

2 Marks Questions

6. If f(x) = x^2g(x) and g (1) = 6, g´(1) =3. find value of f´(1). ***[Ans. 15]***

7. What is the distance between the planes $2x + 2y - z + 2 = 0$ and $4x + 4y - 2z + 5 = 0$.

[Ans. $\frac{1}{6}$]

8. Write the line $2x = 3y = 4z$ in vector form.

9. Find the equation of the line passing through point $(1,0,2)$ having direction ratios $3, -1, 5$. prove that this line passes through $(4, -1, 7)$. **[Ans. $\frac{x-1}{3} = \frac{y-0}{-1} = \frac{z-2}{5}$]**

10. Find the equation of the line parallel to the line $\frac{x-2}{3} = \frac{y+1}{1} = \frac{z-7}{9}$ and passing through the point $(3, 0, 5)$

[Ans. $\frac{x-3}{3} = \frac{y-0}{1} = \frac{z-5}{9}$]

4 Marks Questions

11. Let N be the set of all natural numbers. A relation R be defined on N× N by $(a,b)R\ (c,d) \Leftrightarrow a + b = b + c$. Show that R is equivalence relation.

Solution: (i) Since a + b = b + a ∴ $(a,b)R(a,b)$.

∴ R is reflexive.

(ii) (a, b) R (c, d) ⇒ a + d = b + c
⇒ b + c = a + d
⇒ c + b = d + a
⇒ (c, d) R (a, b)

∴ R is symmetric

(iii) (a, b) R (c, d) and (c, d) R (e, f)
⇒ a + d = b + c and c + f = d + e
⇒ a + d + c + f = b + c + d + e
⇒ a + f = b + e
⇒ (a, b) R (e, f)

∴ R is transitive. Thus R is an equivalence relation on N × N.

12. Let A = N× N, and let * be a binary operation on A defined by (a, b) * (c, d) = (ad + bc, bd) for all (a, b), (c, d) ∈ $N \times N$.

Show that (i) '*' is commutative on A.
(ii) '*' is associative on A.
(iii) '*' A has no identity element.

Solution: (i) For any (a, b), (c, d) ∈ $N \times N$, we have
(a, b) * (c, d) = (ad + bc, bd)
And, (c, d) * (a, b) = (cb + da, bd)

Since addition and multiplication are commutative on N.

Therefore ad + bc = cb + da and bd = db

⇒ (ad + bc, bd) = (cb + da, db)

⇒ (a, b) * (c, d) = (c, d) * (a, b).

(ii) For any (a, b), (c, d), (e, f) ∈ A, we have

[(a, b) * (c, d)] * (e, f) = (ad + bc, bd) * (e, f)

= ((ad + bc)f + (bd)e, (bd)f)

= (adf + bcf + bde, bdf) (1)

[By comm. and Assoc. of add. and mult. on N]

And (a, b) * {(c, d) * (e, f)} = (a, b) * (cf + de, df)

= (a(df) + b(cf + de), b(df))

= (adf + bcf + bde, bdf) (2)

From (1) and (2) , we get {(a, b) * (c, d)} * (e, f)

= (a, b) * {(c, d) * (e, f)} for all (a, b), (c, d), (e, f) ∈ $N \times N = A$

So, * is associative on A.

(iii) If possible, let (x, y) be the identity element in A.

Then (a, b) * (x, y) = (a, b) for all a, b ∈ N

⇒ (ay +bx, by) = (a, b) for all a, b ∈ N

⇒ ay + bx = a and by = b for all a, b ∈ N

⇒ x = 0, y = 1.

But, $0 \notin N. Therefore, (0,1) \notin N \times N = A$.

So, there is no identity element in A with respect to '*'.

13. Prove that

$sin^{-1}\frac{4}{5} + sin^{-1}\frac{5}{13} + sin^{-1}\frac{16}{65} = \frac{\pi}{2}$

Solution: L.H.S = $\sin^{-1}\left(\frac{4}{5}.\sqrt{1-\frac{25}{169}} + \sqrt{1-\frac{16}{25}}.\frac{5}{13}\right) + sin^{-1}\frac{16}{65}$

$= \sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right) + sin^{-1}\frac{16}{65}$

$= \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65}$

$= \sin^{-1}\left(\frac{63}{65}.\sqrt{1-\left(\frac{16}{65}\right)^2} + \frac{16}{65}.\sqrt{1-\left(\frac{63}{65}\right)^2}\right)$

$= \sin^{-1}\left(\frac{63}{65}.\frac{63}{65} + \frac{16}{65}.\frac{16}{65}\right)$

$= \sin^{-1}\left(\frac{63^2+16^2}{65^2}\right) = \sin^{-1}\left(\frac{65^2}{65^2}\right) = sin^{-1}1 = \frac{\pi}{2} = R.H.S.$

14. If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, $show\ that\ A^2 - 5A + 7I = 0. Use\ this\ result\ to\ find\ A^{-1}$.

15. If A = $\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$, $then\ prove\ by\ principle\ of\ Mathematical\ Induction\ that\ A^n = \begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix}$, where n∈ N.

16. Express the following matrix as the sum of a symmetric and a skew symmetric matrix. $\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$

***Solution*:** For any square matrix A,

A = $\frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

Where ½ (A + A') is a symmetric matrix and ½ (A – A') is skew – symmetric matrix.

Given A = $\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ $\therefore A' = \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$

∴ $\frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 1+1 & 3-6 & 5-4 \\ -6+3 & 8+8 & 3+6 \\ -4+5 & 6+3 & 5+5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}$

$$= \begin{bmatrix} 1 & \frac{9}{2} & \frac{1}{2} \\ -\frac{3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix} \qquad \text{.........(1)}$$

This matrix is a symmetric matrix

Again $\frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 1-1 & 6-3 & 5+4 \\ -6-3 & 8-8 & 3-6 \\ -4-5 & 6-3 & 5-5 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ -\frac{9}{2} & 0 & -\frac{3}{2} \\ -\frac{9}{2} & \frac{3}{2} & 0 \end{bmatrix} \qquad \text{..........(2)}$$

This matrix is a skew-symmetric matrix,

Now A $= \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & 8 & \frac{9}{2} \\ \frac{1}{2} & \frac{9}{2} & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{9}{2} & \frac{9}{2} \\ -\frac{9}{2} & 0 & -\frac{3}{2} \\ -\frac{9}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

= Sum of a symmetric and a skew-symmetric matrix.

17. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = (abc)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) = (bc + ca + ab + abc).$$

Solution: Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Then, $\Delta = (abc).\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$

[Taking a, b, c common from R_1, R_2, R_3 respectively]

$$= (abc).\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$\left[Applying\ R_1 \to R_1 + R_2 + R_3\ and\ taking \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)\ common\ from\ R_1\right]$

$$= (abc).\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & \frac{1}{b} \\ -1 & -1 & \frac{1}{c}+1 \end{vmatrix} \qquad [C_1 \to (C_1 - C_3)\ and\ C_2 \to (C_2 - C_3)]$$

$$= (abc).\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right).(1).\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \qquad [Expanding\ along\ R_1$$

$$= (abc).\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right).1$$

$$= (abc).\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right) = (bc + ca + ab + abc)$$

18. If y = $\tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right), find\ \frac{dy}{dx}$

Solution: Here y = $\tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$

Put $x^2 = \cos\theta$

$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$

$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}\right]$

$= \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$

$= \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x^2$

Diff. w.r.t. x, we get

$\frac{\partial y}{\partial x} = 0 - \frac{1}{2}.\frac{d}{dx}(\cos^{-1}x^2)$

$= -\frac{1}{2}.\frac{-1}{\sqrt{1-(x^2)^2}}.\frac{d}{dx}(x^2)$

$= \frac{1}{2}.\frac{1}{\sqrt{1-x^4}}.2x = \frac{x}{\sqrt{1-x^4}}$

19. If $x = a(\theta + \sin\theta), y = a(1-\cos\theta), find\ \frac{d^2y}{dx^2}\ at\ \theta = \frac{\pi}{2}$.

Solution: Given, $x = a(\theta + \sin\theta) \Rightarrow \frac{dx}{d\theta} = a(1+\cos\theta)$ (1)

and $y = a(1-\cos\theta) \Rightarrow \frac{dy}{d\theta} = a\sin\theta$ (2)

$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$= \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right)$

$= \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right).\frac{d\theta}{dx}$

$= \frac{1}{2}sec^2\frac{\theta}{2}/\frac{dx}{d\theta}$

$= \frac{\frac{1}{2}sec^2\frac{\theta}{2}}{a(1-\cos\theta)}$

At $\theta = \frac{\pi}{2}, \frac{d^2y}{dx^2} = \frac{1}{2}sec^2\frac{\pi}{4}.\frac{1}{a\left(1+\cos\frac{\pi}{2}\right)} = \frac{1}{a}$

20. Verify Rolle's theorem for the function $f(x) = 2x^3 + x^2 - 4x - 2, where -\frac{1}{2} \le x \le \sqrt{2}$.

Solution: Given, $f(x) = 2x^3 + x^2 - 4x - 2$ (1)

f'(x) = $6x^2 + 2x - 4$ (2)

Clearly f'(x) is finite and unique for all x and hence f(x) is differentiable and continuous at all x, (Alternatively as f(x) is polynomial in x, it is differentiable and continuous everywhere) Therefore,

(i) f(x) is continuous at all x and hence also continuous in $\left[-\frac{1}{2}, \sqrt{2}\right]$

(ii) f(x) is differentiable in $\left(-\frac{1}{2}, \sqrt{2}\right)$

(iii) Also from (1), $f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{8}\right) + \frac{1}{4} - 4\left(-\frac{1}{2}\right) - 2 = 0\ and$

$$f(\sqrt{2}) = 2.2\sqrt{2} + 2 - 4\sqrt{2} - 2 = 0\ \therefore f\left(-\frac{1}{2}\right) = f(\sqrt{2})$$

Hence all conditions of Rolle's theorem are satisfied for f(x) in $\left[-\frac{1}{2}, \sqrt{2}\right]$

Now from (2), f'(c) = 0

$\Leftrightarrow \quad 6c^2 + 2c - 4 = 0 \Leftrightarrow 3c^2 + c - 2 = 0$

$\Leftrightarrow \quad 3c^2 + 3c - 2c - 2 = 0 \Leftrightarrow (c+1)(3c-2) = 0$

$\Leftrightarrow \quad$ c = -1, 2/3

But $\quad -\frac{1}{2} < c < \sqrt{2} \quad \therefore c = \frac{2}{3}$

Thus we get at least one $c\left(=\frac{2}{3}\right), where\ -\frac{1}{2} < c < \sqrt{2}\ such\ that\ f'(c) = 0$

Thus Rolle's theorem has been verified.

21. Verify Lagrange's mean value theorem for the function $f(x) = \sqrt{x^2 - 4}\ in\ the\ interval\ [2, 4]$.

Solution: $\quad f(x) = \sqrt{x^2 - 4}\ for\ x \in [2,4] \qquad (1)$

From (1), $\quad f(x) = \frac{x}{\sqrt{x^2-4}}$

Clearly f(x) is continuous in [2, 4] and derivable in (2, 4).

⇒ Conditions of Lagrange's mean value theorem are satisfied for f(x) in [2, 4]

⇒ $\ for\ all\ \ c \in (2,4)\ such\ that$

$$f(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(c)}{4-2}$$

⇒ $\quad \frac{c}{\sqrt{x^2-4}} = \frac{1}{2}.\left[\sqrt{4^2-4} - 0\right] = \sqrt{3}$

⇒ $\quad c^2 = 3(c^2-4)$

⇒ $\quad 2c^2 = 12 \Rightarrow c = \sqrt{6} \qquad \left[c = -\sqrt{6} does\ not\ lie\ in\ (2,4)\right]$

Hence $\quad$ c = $\sqrt{6} \in (2,4)$.

Thus Lagrange's mean value theorem has been verified.

22. Find the intervals in which the function f given by f(x) = $2x^3 - 21x^2 + 36x - 40$ is (i) strictly increasing, (ii) strictly decreasing.

Solution: Here, f(x) = $2x^3 - 21x^2 + 36x - 40$

⇒ $\quad f'(x) = 6x^2 - 42x + 36$

$\quad = 6(x^2 - 7x + 6)$

$\quad = 6(x - 1)(x - 6)$

Sign scheme for f'(x) i.e., for (x – 1)(x – 6) is

$-\infty \longleftarrow\!\!\mid\!\!\longrightarrow\!\!\mid\!\!\longrightarrow \quad \infty$

+ve $\quad$ 1 $\quad$ -ve $\quad$ 6 $\quad$ +ve

Put x = 0

From sign scheme for f'(x), it is clear that

(i) $\quad$ F'(x) > 0 in (∞, 1) and in (6, ∞), therefore, f is strictly increasing in (-∞, 1) and in (6, ∞).
Here f is also strictly increasing in (-∞, 1] and [6, ∞) as it is continuous at x = 1 and x = 6.

(ii) $\quad$ F'(x) < 0 in (1,), therefore, f is strictly decreasing in (1,6).
Here f is also strictly decreasing in [1,6].

23. Find the approximate value of $(255)^{\frac{1}{4}}$ using differentials.

Solution: we have to find the approximate value of $(255)^{\frac{1}{4}}$, therefore

We take $\quad f(x) = x^{1/4} \qquad (1)$

$\therefore \quad f'(x) = \left(\frac{1}{4}x\right)^{-\frac{3}{4}} = \frac{1}{4x^{\frac{3}{4}}} \qquad (2)$

Now $\quad f(x + \delta x) = f(x) + \delta x . f'(x)\ (approximately)$

⇒ $\quad (x + \delta x)^{\frac{1}{4}} = x^{\frac{1}{4}} + \frac{1}{4x^{\frac{3}{4}}} . \delta x \qquad (3)$

We know the value of $(256)^{\frac{1}{4}}$ which is equal to 4.

Putting $\quad$ x = 256, $x + \delta x = 255\ so\ that\ \delta x = -1\ in\ (3), we\ get$

$$(255)^{\frac{1}{4}} = (256)^{\frac{1}{4}} + \frac{1}{4(256)^{\frac{3}{4}}}(-1)$$

$$= 4 + \frac{1}{4\times4^3}(-1)$$

$$= 4 - \frac{1}{256} = \frac{1023}{256} = 3.9961.$$

24. Water is running out of a conical funnel at the rate of 5 cm^3/sec. if the radius of the base of the funnel is 10 cm and altitude is 20 cm, find the rate at which the water level is dropping when it is 5 cm from the

top. **[Ans. $\frac{4}{45\pi}$ $cm/$**
sec]

25. Find the intervals in which the function $f(x) = sin^4x + cos^4x, 0 \le x \le \frac{\pi}{2}$ is increasing or decreasing.

[Increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ decreasing in $\left(0, \frac{\pi}{4}\right)$]

26. Find the value of $\sqrt{0.037}$. ***[0.1925]***

27. Evaluate: $\int \frac{2x-3}{(x^2-1)(2x+3)}\, dx$

Solution:- Let $I = \int \frac{2x-3}{(x^2-1)(2x+3)}\, dx$

Now, $\frac{2x-3}{(x^2-1)(2x+3)} = \frac{2x-3}{(x-1)\,(x+1)(2x+3)}$

$= \frac{-1}{10\,(x-1)} + \frac{5}{2(x+1)} - \frac{24}{5(2x+3)}$ [on resolving into partial fractions]

$\therefore$ $I = -\frac{1}{10}\int \frac{dx}{x-1} + \frac{5}{2}\int \frac{dx}{x+1} - \frac{24}{5}\int \frac{dx}{2x+3}$

$= -\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + c$

28. Evaluate : $\int \frac{2x+1}{2x^2+4x-3}\, dx$

Solution:- Let $I = \int \frac{2x+1}{2x^2+4x-3}\, dx$

Let $2x+1 = A\frac{d}{dx}(2x^2 + 4x - 3) + B = A(4x+3) + B$

Equation the coefficient of similar power of x, we get

$4A = 2 \Rightarrow A = \frac{1}{2}$

and $4A + B = 1 \Rightarrow B = -1$

Now, $I = \int \frac{\frac{1}{2}(4x+4)-1}{2x^2+4x-3}\, dx$

$= \frac{1}{2}\int \frac{4x+4}{2x^2+4x-3}\, dx - \int \frac{dx}{2x^2+4x-3} + c$

$= \frac{1}{2} I_1 - I_2 + c$

Where, $I_1 = \int \frac{4x+4}{2x^2+4x-3}\, dx$ *and* $I_2 = \int \frac{dx}{2x^2+4x-3}$

To find : put $2x^2 + 4x - 3 = t \Rightarrow (4x+4)dx = dt$

Now, $I_1 = \int \frac{dt}{t} = \log 1t = \log|2x^2+4x-3|$

Again, $I_2 = \int \frac{dx}{2x^2+4x-3} = \frac{1}{2}\int \frac{dx}{x^2+2x-\frac{3}{2}}$

$= \frac{1}{2}\int \frac{dx}{(x+1)^2 - \left(\sqrt{\frac{5}{2}}\right)^2}$

$= \frac{1}{2}\cdot\frac{1}{2\sqrt{\frac{5}{2}}}\log\left|\frac{x+1-\sqrt{\frac{5}{2}}}{x+1+\sqrt{\frac{5}{2}}}\right|$

$= \frac{1}{2\sqrt{10}}\log\left|\frac{x+1-\sqrt{\frac{5}{2}}}{x+1+\sqrt{\frac{5}{2}}}\right|$

From (1), (2) and (3), we get

$$I = \frac{1}{2}\log|2x^2+4x-3| - \frac{1}{2\sqrt{10}}\log\left|\frac{x+1-\sqrt{\frac{5}{2}}}{x+1+\sqrt{\frac{5}{2}}}\right| + c$$

29. Evaluate : $\int_0^{\pi/2}\left(\sqrt{\tan x} + \sqrt{\cot x}\right)\, dx$

Solution:- $\int_0^{\pi/2}\left(\sqrt{\tan x} + \sqrt{\cot x}\right)\, dx = \int_0^{\pi/2}\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$

$= \int_0^{\pi/2} \frac{\sin x+\cos x}{\sqrt{\sin x \cos x}}\, dx = \sqrt{2} \quad \int_0^{\pi/2} \frac{\sin x+\cos x}{\sqrt{2 \sin x \cos x}}\, dx$]

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

When $x = 0, t = \sin 0 - \cos 0 = -1$ *and*

When $x = \frac{\pi}{2}, t = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = 1 - 0 = 1$

Now, $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \sqrt{2} \quad \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\ [sin^{-1}]_{-1}^{1}$

$= \sqrt{2}\,[sin^{-1}(1) - sin^{-1}(-1)] = \sqrt{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \qquad = \sqrt{2}\,\pi$

30. Evaluate: $\int_0^{\pi/2} \frac{\sin 2x}{sin^4 x + cos^4 x}\, dx$

Solution:- Let $I = \int_0^{\pi/2} \frac{\sin 2x}{sin^4 x + cos^4 x}\, dx = \int_0^{\pi/2} \frac{\sin 2x}{sin^4 x + (1 - sin^2 x)^2}\, dx$

Let $sin^2 x = t. then, 2 \sin x \cos x\, dx = dt \implies \sin 2x\ dx = dt$

Also $x = 0 \implies t = sin^2 0 = 0\ and\ x = \frac{\pi}{2} \implies t\ sin^2 \frac{\pi}{2} = 1$

Now, $I = \int_0^1 \frac{dt}{t^2 + (1-t)^2}$

$= \int_0^1 \frac{dt}{2t^2 - 2t + 1}$

$= \frac{1}{2}\int_0^1 \frac{dt}{t^2 - t + \frac{1}{2}}$

$= \frac{1}{2}\int_0^1 \frac{dt}{t^2 - t + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}} \frac{1}{2}\int_0^1 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$

$= \frac{1}{2} . \frac{1}{\left(\frac{1}{2}\right)} \left[tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1 \qquad = [tan^{-1}(2t-1)]_0^1 = tan^{-1} 1 - tan^{-1}(-1) =$

$\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

31. Evaluate : $\int_0^{\pi/2} \log \sin x dx$

Solution:- Let $I = \int_0^{\pi/2} \log \sin x dx$ …..(1)

Then, $I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right)\, dx$

$= \int_0^{\pi/2} \log \cos x dx$ …..(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\ dx = \int_0^{\pi/2} \log(\sin x \cos x) dx$

$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)\, dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$

$= \int_0^{\pi/2} \log \sin 2x dx - \log 2 \int_0^{\pi/2} dx$

$= \int_0^{\pi/2} \log \sin 2x dx - \frac{\pi}{2} \log 2$ …..(3)

Now, $\int_0^{\pi/2} \log \sin 2x dx = \frac{1}{2}\int_0^{\pi} \log \sin z dz$

$= \frac{1}{2} 2. \int_0^{\pi/2} \log \sin z dz$

$= \int_0^{\pi/2} \log \sin z dz = \int_0^{\pi} \log \sin x dx = I$

From (3) $2I = I - \frac{\pi}{2} \log 2 \qquad I = -\frac{\pi}{2} \log 2$

Note:

(1) $\int_0^{\pi/2} log\ cos\ x dx = \int_0^{\pi/2} log\ sin\ x dx = -\frac{\pi}{2}\ log\ 2$

(2) $\int_0^{\pi/2} \log cosec\ x dx = -\int_0^{\pi/2} \log \sin x dx = \frac{\pi}{2} \log 2$

$\int_0^{\pi/2} \log \sec x dx = -\int_0^{\pi/2} \log \cos x dx = \frac{\pi}{2} \log 2$

32. Evaluate : $\int_1^4 (x^2 - x) dx$as the limit of a sum.

Solution:- $\int_a^b f(x)\, dx = \lim_{h\to 0} h \sum_{r=1}^{n} f(a + rh)$ …..(1)

Where, $nh = b - a$ and $n \to \infty$

Here, $f(x) = x^2 - x, a = 1, b = 4$

∴ $nh = b - a = 4 - 1 = 3$

Now, $f(a + rh) = (a + rh)^2 - (a + rh)$

$= (1 + rh)^2 - (1 + rh)$

$= 1 + 2rh + r^2h^2 - 1 - rh$

$= h^2r^2 + rh$

From (1), $\int_1^4 (x^2 - x)\, dx = \lim_{h\to 0} \sum_{r=1}^{n} h\,(h^2r^2 + rh)$

$= \lim_{h \to 0} \sum_{r=1}^{n} (h^3\, r^2 + h^2 r)$

$= \lim_{h \to 0} (h^3 \sum_{r=1}^{n} r^2 + h^2\ \sum_{r=1}^{n} r)$

$= \lim_{h \to 0} [h^3\ (1^2 + 2^2 + \ \ldots\ldots + n^2) + h^2(1 + 2 + \cdots \ldots. + n)]$

$= \lim_{h \to 0} \left[h^3 \frac{n(n+1)\,(2n+1)}{6} + \ h^2\ \frac{n(n+1)}{2}\right]$

$= \lim_{h \to 0} \left[\frac{nh(nh+h)(2nh+h)}{6} + \frac{nh\,(nh+h)}{2}\right]$

$= \lim_{h \to 0} \left[\frac{3(3+h)(6+h)}{6} + \frac{3(3+h)}{2}\right]$

$= \frac{3\times3\times6}{6} + \frac{3\times3}{2} \qquad = 9 + \frac{9}{2} = \frac{27}{2}$

33. If $\vec{a} = 5\hat{\imath} - \hat{\jmath} + 7\hat{k}$ and $\vec{b} = \hat{\imath} - \hat{\jmath} + \lambda\hat{k}$, find λ such that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

Solution:- $\vec{a} + \vec{b} = (5\hat{\imath} - \hat{\jmath} + 7\hat{k}) + (\hat{\imath} - \hat{\jmath} + \lambda\hat{k})$

$= 6\hat{\imath} - 2\hat{\jmath} + (7 + \lambda)\,\hat{k}$

$\vec{a} + \vec{b} = 4\hat{\imath} + 0\hat{\jmath} + (7 + \lambda)\hat{k}$

Now $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal

$\Rightarrow \quad (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$

$\Rightarrow \quad [6\hat{\imath} - 2\hat{\jmath} + (7 + \lambda)\hat{k}].[4\hat{\imath} + 0\hat{\jmath} + (7 - \lambda)\,\hat{k}] = 0$

$\Rightarrow \quad 6 \times 4 - 2 \times 0 + (7 + \lambda).(7 - \lambda) = 0$

$\Rightarrow \quad 24 + 49 - \lambda^2 = 0$

$\Rightarrow \quad \lambda^2 = 73 \Rightarrow \lambda = \pm\sqrt{73}$

34. Find a unit vector perpendicular to the plane of two vectors. $\vec{a} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$

Solution:- By definition of vector product, $\vec{a} \times \vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Therefore, unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

$= \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{cross\ product}{Modulus\ of\ cross\ product}$ …..(1)

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix} = (1 - 6)\hat{\imath} - (-1 - 4)\hat{\jmath} + (3 + 2)\,\hat{k}$

$= -5\hat{\imath} + 5\hat{\jmath} + 5\hat{k}$

∴ $|\vec{a} \times \vec{b}| = \sqrt{(-5)^2 + (5)^2 + (5)^2} = \sqrt{75} = 5\sqrt{3}$

Hence form (1), the required unit vector

$= \frac{-5\hat{\imath}+5\hat{\jmath}+5\hat{k}}{5\sqrt{3}}$

$= \frac{\hat{\imath}}{\sqrt{3}} + \frac{\hat{\jmath}}{\sqrt{3}} + \frac{\hat{k}}{\sqrt{3}}$

35. Two dice are rolled once. Find the probability that:

(i) The number on two dice are different

(ii) The total of numbers on the two dice is at least 4

Solution:-

Total number of cases = $6 \times 6 = 36$

(i) When the number on the two dice are different,

The number of favorable cases = $6 \times 5 = 30$

∴ The reqd. prob $= \frac{30}{36} = \frac{5}{6}$

(ii) $P\ (X\ \geq 4)$

Where $X = toatl\ of\ the\ number\ on\ the\ two\ dice$

$= 1 - [P\ (X = 2) +\ P\ (X = 3)]$

$= 1 - P(X = 2) -\ P\ (X = 3)$

$= 1 - \frac{1}{36} - \frac{2}{36}$

(There is only one favorable case for X = 2, namely (1, 1) and two favorable cases for X = 3, namely (1, 2)
and (2, 1))

$= 1 - \frac{3}{36} = 1 - \frac{1}{12} = \frac{11}{12}$

36. From a lot of 30 bulbs, which includes 6 defective bulbs? A sample of 3 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Solution:-

There are 6 defective bulbs and 24 non-defective bulbs. Let X denotes the random variable: "the number of defective bulbs". Then X can take values 0, 1, 2 or 3.

Let D_1, D_2, D_3 be the events of drawing a defective bulb in first, second and third draws respectively.

Since bulbs are replaced, therefore $P\ (D_1) = P\ (D_2) = P\ (D_3) = \frac{6}{30} = \frac{1}{5}$

Clearly D_1, D_2, D_3 are independent events probability that

Now, $P\ (X = 0) = P\ (\bar{D}_1\ \bar{D}_2\bar{D}_3)$

$= P\ (\bar{D}_1)\ P\ (\bar{D}_2)\ P\ (\bar{D}_3)$

$= \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} = \frac{64}{125}$

$P\ (X = 1) = P\ (D_1\ \bar{D}_2\bar{D}_3\ \ or\ \ \bar{D}_1 D_2\bar{D}_3\ or\ \bar{D}_1\bar{D}_2 D_3)$

$= P\ (D_1\bar{D}_2\bar{D}_3) +\ P\ (\bar{D}_1 D_2\bar{D}_3) +\ P\ (\bar{D}_1\bar{D}_2 D_3)$

$= P\ (D_1)\ P\ (\bar{D}_2)\ P(\bar{D}_3) +\ P(\bar{D}_1)\ P(D_2)\ P\ (\bar{D}_3)$

$= \frac{1}{5} \times \frac{4}{5} \times \frac{4}{5} + \frac{4}{5} \times \frac{1}{5} \times \frac{4}{5} + \frac{4}{5} \times \frac{4}{5} \times \frac{1}{5}$

$= \frac{16}{125} + \frac{16}{125} + \frac{16}{125} = \frac{48}{125}$

$P\ (X = 2) = P\ (D_1 D_2\bar{D}_3\ or\ D_1\bar{D}_2 D_3\ or\ \bar{D}_1 D_2 D_3)$

$= P\ (D_1 D_2\bar{D}_3) +\ P\ (D_1\bar{D}_2 D_3) +\ P\ (\bar{D}_1 D_2 D_3)$

$= P\ (D_1)\ P\ (D_2)\ P\ (\bar{D}_3) +\ P\ (D_1)\ P\ (\bar{D}_2)\ P\ (D_3) +\ P\ (\bar{D}_1)\ P\ (D_2)\ P\ (D_3)$

$= \frac{1}{5} \times \frac{1}{5} \times \frac{4}{5} + \frac{1}{5} \times \frac{4}{5} \times \frac{1}{5} + \frac{4}{5} \times \frac{1}{5} \times \frac{1}{5}$

$= \frac{4}{125} + \frac{4}{125} + \frac{4}{125} = \frac{12}{125}$

$P\ (X = 3) = P\ (D_1\ D_2 D_3)$

$= P\ (D_1)\ P\ (D_2)\ P\ (D_3)\ \ = \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5} = \frac{1}{125}$

∴ The **required probability** distribution of X is:

x	0	1	2	3
$P(x)$	$\frac{64}{125}$	$\frac{48}{125}$	$\frac{12}{125}$	$\frac{1}{125}$

37. 10 coins are tossed what is the probability that exactly 5 heads appear? Also find the probability of getting at least 8 heads.

Solution:-

Let E = the event of occurrence of head on one coin

Then $P = P\ (E) = \frac{1}{2}\ \ and\ \ \ q = 1 - p = \frac{1}{2}$

Here $n = 10\ (since\ 10\ coins\ have\ been\ tossed)$

Now $P\ (r) =$ probability that events E will occure exactly r times in 5 trials

$= n_{C_r}\, P^r\, q^{n-r}$

$\therefore$ **Required probability** $= 10_{C_5}\, p^5 q^5 = \frac{\lfloor 10}{\lfloor 5 \lfloor 5} \cdot \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 = \frac{63}{256}$

Second part: Required probability

$= P\,(8) + P\,(9) + P\,(10)$

$= 10_{C_8}\, p^8 q^2 + 10_{C_9}\, p^9 q^1 + 10_{C_{10}}\, p^{10} q^0$

$= 45\, \left(\frac{1}{2}\right)^8 . \left(\frac{1}{2}\right)^2 + 10\, \left(\frac{1}{2}\right)^9 . \frac{1}{2} + 1 . \left(\frac{1}{2}\right)^{10}$

$= 56. \left(\frac{1}{2}\right)^{10} = \frac{7}{2^7} = \frac{7}{128}$

38. The probability that student entering a university will graduate is 0.4. find the probability that out of 3 students of the university:
(i) None will graduate.
(ii) Only one will graduate.
(iii) All will graduate.

Solution:- Let X be the number of students who graduate.

$P =$ probability that a student graduate = 0.4

$\Longrightarrow \quad q = 1 - p = 1 - 0.4 = 0.6 \qquad and \quad n = 3.$

Here : $(q + p)^n = (0.6 + 0.4)^3$

(i) $P\,(none\ will\ graduate\,)$

$= P\,(X = 0) = n_{C_0}\, q^n p^0$

$= 3_{C_0}\, q^3 = (0.6)^3 = 0.216$

(ii) $P\,(only\ one\ will\ graduate)$

$= P\,(X = 1) = n_{C_1}\, q^{n-1} p^1$

$= 3_{C_1}\, q^2\, p = 3\,(0.6)^2\,(0.4) = 0.432$

(iii) $P\,(All\ will\ graduate)$

$= P\,(X = 3) = n_{C_3}\, q^{n-3} p^3 = n_{C_3}\, q^0 p^3$

$= (0.4)^3 = 0.064$

39. Differentiate $\left(x + \frac{1}{x}\right)^x + x^{\left(1 + \frac{1}{x}\right)}$ w.r.t. x.

[Ans. $\boldsymbol{\left(x + \frac{1}{x}\right)^x \left[\frac{x^2-1}{x^2+1} + \log\left(x + \frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)} \left[\frac{1}{x}\left(1 + \frac{1}{x}\right) - \frac{1}{x^2}\log x\right]}$***]***

40. If $y = A\left(x + \sqrt{1 + x^2}\right)^n$, prove that $(x^2 + 1)\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} - n^2 y = 0$

41. Find the value of 'a' for which the function f defined as $f(x) = \begin{cases} a \sin\frac{\pi}{2}\,(x+1), & if\ x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & if\ x > 0 \end{cases}$ is continuous at $x = 0$.

Ans. $L.H.L._{x \to 0} = \lim_{h \to 0}\, a \sin\frac{\pi}{2}\,\{-h + 1\}$

$= \lim_{h \to 0}\, a \sin\left\{\frac{\pi}{2} - \frac{\pi h}{2}\right\}$

$= \lim_{h \to 0}\, a . \cos\frac{\pi h}{2} = a . \cos 0 - a.$

$R.H.L._{x \to 0} = \lim_{h \to 0} \frac{\tan h - \sin h}{h^3}$

$= Lim_{h \to 0} \frac{\tan h}{h} . \frac{(1 - \cos h)}{h^2}$

$= Lim_{h \to 0} \frac{\tan h}{h} . \frac{2\sin^2\left(\frac{h}{2}\right)}{4.\left(\frac{h}{2}\right)^2}$

$= \frac{1}{2}\lim_{h \to 0}\left(\frac{\tan h}{h}\right)\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 = \frac{1}{2}$

For continuity at x = 0:

$L.H.L_{x=0} = R.H.L._{x=0} = f(0) = a$

$a = \frac{1}{2} = a \Rightarrow a = \frac{1}{2}$

6 Marks Questions

42. Consider $f: R \to [-5, \infty)$ $given\ by\ f(x) = 9x^2 + 6x - 5, where\ R^+\ is\ the\ set\ of\ all\ non-negative\ real\ numbers$. show that f is invertible with $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$.

Solution: Given $f: R \to [-5, \infty)$ $given\ by\ f(x) = 9x^2 + 6x - 5$ ………(i)

To test whether f is one-one: Let $x_1, x_2 \in R_+$ such that $f(x_1) = f(x_2)$

Now $f(x_1) = f(x_2) \Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_1^2 + 6x_2 - 5$

$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow x_1 - x_2 = 0$ $[\because x_1, x_2 \in R_+ \therefore 9(x_1 + x_2) + 6 \neq 0]$

$\Rightarrow x_1 = x_2$

Hence f is one-one.

To test whether f is onto: Let y be an arbitrary element of [-5, ∞)

Let $f(x) = y$

Now $y = f(x) \Rightarrow y\ 9x^2 + 6x - 5$

$\Rightarrow 9x^2 + 6x - (5 + y) = 0$

$\Rightarrow x = \frac{-6 \pm \sqrt{36+36(5+y)}}{18}$

$\Rightarrow x = \frac{-6 \pm 6\sqrt{6+y}}{18}$

$\Rightarrow x = \frac{-1 \pm \sqrt{6+y}}{3}$

$\Rightarrow x = \frac{-1+\sqrt{6+y}}{3} = \frac{\sqrt{6+y}-1}{3}$ $[\because y \in [-5, \infty) \therefore x \geq 0]$

$\Rightarrow x \in domain\ R_+$

Hence f is onto.

To find f^{-1}:

$(fof^{-1})x = x$

$f(f^{-1}(x)) = x$

$g(f^{-1}(x))^2 + 6f^{-1}(x) - 5 - x = 0$

$f^{-1}(x) = 6 \pm \sqrt{36 + \cdots}$

$f^{-1}(x) = \frac{\sqrt{6+x}-1}{3}$

$f^{-1}(y) = \frac{\sqrt{6+y}-1}{3}$

43. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ to solve the system of equations $x - y + 2z = 1,\ 2y - 3z = 1, 3x - 2y + 4z = 2.$

44. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ $and\ B$ $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$; find AB and hence solve the equations

$2x - y + z = -1$

$-x + 2y - z = 4.$

$x - y + 2z = -3$

45. A square piece of tin of side 18 cm is to be made into a box without a top by cutting a square piece from each corner and folding up the flaps. What should be the side of the square to be cut off, so that the volume of the box be maximum? Also find the maximum volume of the box.

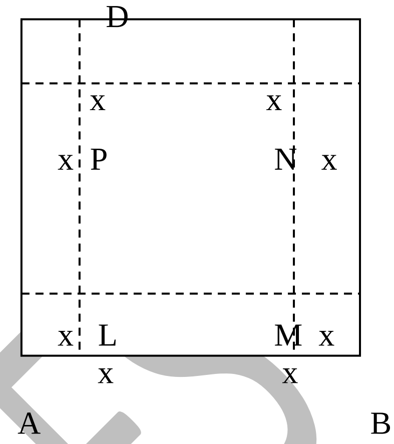
Solution: Let the side of the square piece cut from each corner of the given square plate (side = 18 cm) be x cm. then the open box has dimensions

18 – 2x, 18 – 2x, x (in cm)

$\therefore$ V = Volume of the open box

$= (18 - 2x)^2.x$

$\therefore$ $V = 324x - 72x^2 + 4x^3$ (1)



$\Rightarrow$ $\frac{dv}{dx} = 324 - 144x + 12x^2$ (2)

And $\frac{d^2V}{dx^2} = -144 + 24x$ (3)

For maximum or minimum,

$$\frac{dv}{dx} = 0 \Rightarrow 324 - 144x + 12x^2 = 0$$

$$\Rightarrow x^2 - 12x + 27 = 0$$

$$\Rightarrow x = 3, 9.$$

Clearly $x \neq 9$.

For x = 3, $\frac{d^2V}{dx^2} = -144 + 24 \times 3 = -72 < 0$

$\therefore$ Volume is maximum , when side of the square cut off is 3 cm.

Max. value of the volume = $(18 - 2 \times 3)^2 \times 3 = 432$ cm^3

46. Show that the right triangle of maximum area that can be inscribed in a circle is can isosceles triangle.
47. A given quantity of metal is to be cast half cylinder with a rectangular box and semicircular ends. Show that the total surface area is minimum when the ratio of the length of cylinder to the diameter of its semicircular ends is $\pi : (\pi + 2)$.
48. Using integration, find the area of $\Delta\ ABC$, whose vertices are A (2, 0), B (4, 5) and C (6, 3).

Solution:- The equation of sides AB is

$$y - 0 = \frac{(5-0)}{(4-2)}\ (x-2)\ or\ y = \frac{5}{2}(x-2) \quad \text{.....(1)}$$

$The\ equation\ of\ sides\ BC\ is$

$$y - 5 = \frac{(3-5)}{(6-4)}\ (x-4)\ or\ y = \ -x + 9 \quad \text{.....(2)}$$

$The\ equation\ of\ sides\ AC\ is$

$$y - 0 = \frac{(3-0)}{(6-2)}\ (x-2)\ or\ \ y = \frac{3}{4}(x-2)$$

We draw perpendicular BP and CQ on the x – axis.

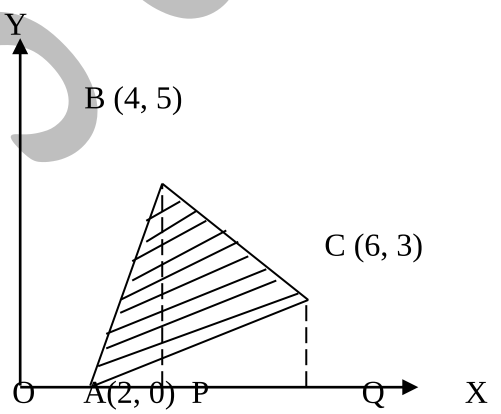
Area of $\Delta\, ABC = ar\ (\Delta\, APB) + \ ar\ (trapezium\ BPQC) - \ ar\ (\Delta\, AQC)$

$$= \int_2^4 Y_{AB}\, dx + \int_4^6 Y_{BC}\, dx - \int_2^6 Y_{AC}\, dx$$

$$= \frac{5}{2}\int_2^4 (x-2)\, dx + \int_4^6 (9-x)\, dx - \frac{3}{4}\int_2^6 (x-2)\ dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2} - 2x\right]_2^4 + \left[9x - \frac{x^2}{2}\right]_4^6 - \frac{3}{4}.\left[\frac{x^2}{2} - \ 2x\right]_2^6$$

$$= \frac{5}{2}[0 - (-2)] + \ (36 - 28) - \frac{3}{4}[6 - (-2)] \qquad = (5 + 8 - 6)\text{sq. units } = 7 \text{ sq. units.}$$



49. Find the area of the region included between the parabolas $y^2 = 4ax\ and\ x^2 = 4ay, where\ a > 0$.

Solution:- The given parabola are $y^2 = 4ax$ .....(1)

And $x^2 = 4ay$ …..(2)

In order to find the points of intersection of the given curves, we solve (1) are (2) simultaneously.

Putting $x = \frac{y^2}{4a}$ $from\ (1)\ in\ (2)\ we\ get$

$\frac{y^4}{16a^2} = 4ay \implies y^4 - 64a^3\, y = 0$

$\implies y\,(y^3 - 64a^3) = 0$

$\implies y = 0\ or, y = 4a$

Now, from (1) $(y = 0 \implies x = 0)\ and\ \left(y = 4a \implies x = \frac{16a^2}{4a} = 4a\right)$

Thus, the points of intersection of the two parabola are O (0, 0) and A (4a, 4a).

Required area OBAO

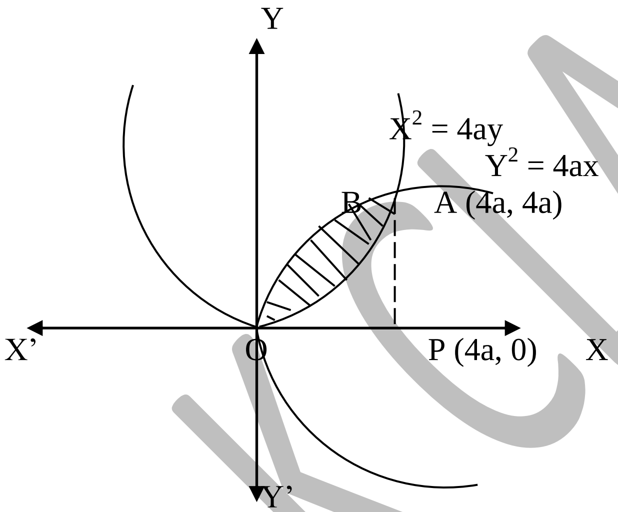
$= \int_0^{4a} (y_1 - y_2)\, dx$

$= \int_0^{4a} y\ \ dx\ \ for\ (y^2 = 4a\,x) - \int_0^{4a} y\ dx\ \ for\ (x^2 = 4ay)$

$= \int_0^{4a} 2\sqrt{ax}\ dx - \int_0^{4a} \frac{x^2}{4a}\ dx$ $= \left[2\sqrt{a}.\frac{2}{3}.x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$

$= \left[\frac{4\sqrt{a}}{3}.(4a)^{3/2} - \frac{1}{12a} \times 64a^3\right]$ $= \left(\frac{32a^2}{3} - \frac{16a^2}{3}\right) = \frac{16a^2}{3}\ sq.units$

Hence, the required area = $\frac{16a^2}{3}$ sq. units.



50. Find the area lying above the x –axis and included between the curves $x^2 + y^2 = 8x\ and\ y^2 = 4x.$

Solution:- The given curves are $x^2 + y^2 = 8x$ …..(1)

And $y^2 = 4x$ …..(2)

The equation (1) can be written as $(x-4)^2 + (y-0)^2 = 4^2$, which represent a circle with centre (4, 0) and radius 4 units.

The equation (2) i.e., $y^2 = 4x$ represent a right handed parabola with vertex (0, 0) and axis y = 0

A rough sketch of the curves is shown in figure solving (1) and (2), we get

$x^2 + 4x = 8x \implies x^2 - 4x = 0 \implies x = 0,4$

When $x = 0, y = 0\ and\ when\ x = 4, y = 4$ (in first quadrant).

Thus the point of intersection of the two curves are (0, 0) and (4,4) above the x – axis.

The required area has been shaded in figure

Required are OPRQO

$= area\ OPQO + area\ QPRQ = \int_0^4 y\ dx\ (for\ parabola) + \int_4^8 y\ dx\ (for\ circle)$

$= \int_0^4 2\sqrt{x}\ dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\ dx$

$\left(\because y^2 = 4x \implies y = 2\sqrt{x}\ and\ (x-4)^2 + y^2 = 4^2 \implies y = \sqrt{4^2 - (x-4)^2}\ in\ the\ first\ quadrant\right)$

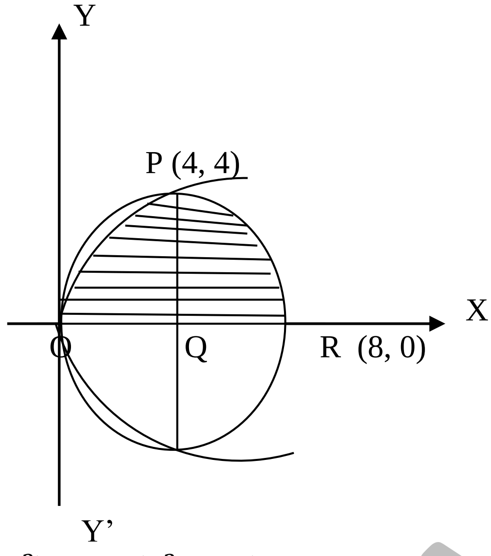
$= 2\left[\frac{2}{3}x^{3/2}\right]_0^4 + \int_0^4 \sqrt{4^2 - t^2}\, dt$

$(In\ 2nd\ integral, put\ x - 4 = t \implies dx = dt, when\ x = 4, t = 0\ and\ when\ x = 8, t = 4)$

$= \frac{4}{3}\left[4^{3/2} - 0\right] + \left[\frac{t\sqrt{4^2-t^2}}{2} + \frac{4^2}{2}\ sin^{-1}\frac{t}{4}\right]_0^4$

$= \frac{4}{3} \times 8 + (0 + 8\, sin^{-1}) - \ (0 + 8\, sin^{-1}0)$

$= \frac{32}{3} + \ 8.\frac{\pi}{2} - 0 = \frac{4}{3}(8 + 3\pi)$ sq. units.



51. Solve : $(3xy + y^2)\, dx + (x^2 + xy)\, dy = 0$

Solution:- The given equation may be written as

$\frac{dy}{dx} = \frac{-\left(3xy+y^2\right)}{(x^2+xy)}$

When x is replaced by $kx\ and\ y\ by\ ky$, R.H.S. of equation (1) remains the same, therefore, equation is a homogeneous differential equation.

Put $y = vx, then \frac{dy}{dx} = v + x\ \frac{dv}{dx}$

Now equation (1) becomes

$v + x\frac{dv}{dx} = \frac{-\left(3vx^2+v^2x^2\right)}{(x^2+vx^2)}$

Or $\quad v + x\ \frac{dv}{dx} = \frac{-\left(3v+v^2\right)}{(1+v)}$

Or $\quad x\ \frac{dv}{dx} = \left[\frac{-\left(3v+v^2\right)}{(1+v)} - v\right]$

Or $\quad x\ \frac{dv}{dx} = \frac{-2\left(2v+v^2\right)}{(1+v)}$

Or $\frac{(1+v)}{(2v+v^2)}\ dv = \ -\frac{2}{x}dx$

Or $\quad \int\frac{(1+v)}{(2v+v^2)}\ dv + \ \int\frac{2}{x}\ dx = \log C$

Or $\quad \frac{1}{2}\log|2v + v^2| + 2\log|x| = \log C$

Or $\quad \log\left|x^2\sqrt{2v + v^2}\right| = \log C$

Or $\quad \log\left|x\ \sqrt{2xy + y^2}\right| = \log C$

Or $\quad x\sqrt{2xy + y^2} = \ \pm C$

Or $\quad x^2\ (2xy + y^2) = C^2$

52. Solve the following differential equation.

$\frac{dy}{dx} + \sec x. y = \tan x\ \left(0 \le x < \frac{\pi}{2}\right)$

Solution:- The given D.E is $\frac{dy}{dx} + \sec x. y = \tan x$ …..(1)

This is a linear D.E. of the form $\frac{dy}{dx} + \ Py = Q$.

Where, $\quad P = \sec x\ and\ Q = \tan x$

Now, I.F.$= e^{\int P\, dx} = e^{\int \sec x\, dx}$

$= e^{\log(\sec x+\tan x)} = \sec x + \tan x$

$\therefore$ The solution of equation (1) is given by

$y\,.(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)\ dx + c$

$= \int(\sec x\ \ \tan x + tan^2 x)\ dx + c$

$= \int \sec x \tan x\ dx + \int sec^2\, xdx - \int 1. dx + c$

$\therefore$ $y\,(\sec x + \tan x) = \sec x + \tan x - x + c$

This is the required solution of the given differential equation.

53. Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}\ and\ \frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find also the point of intersection of these lines.

Solution:- Given lines are

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ …..(1)

And $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$ …..(2)

Any point on line (1) is $P\ (2r+1, 3r+2, 4r+3)$

Any point on line (2) is $Q\ (5\lambda+4, 2\lambda+1, \lambda)$

Lines (1) and (2) will intersect if P and Q coincide for some value of $\lambda\ and\ r$.

$\therefore$ $2r+1 = 5\lambda+4 \implies 2r - 5\lambda = 3$ …..(1)

$3r+2 = 2\lambda+1 \implies 3r - 2\lambda = -1$ …..(2)

$4r+3 = \lambda \implies 4r - \lambda = -3$ …..(3)

Solving (1) and (2), we get $r = -1, \lambda = -1$

Clearly these values of $r\ and\ \lambda\ satisfy\ equation$ (3)

Putting the value of $r, we\ have\ P\ \equiv (-1,-1,-1)$

Hence lines (1) and (2) intersect at $(-1,-1,-1)$.

54. Find the distance of the point (2, 3, 4) from the plane $3x + 2y + 2z + 5 = 0,$ measured parallel to the line $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$.

Solution:- The given plane is

$3x + 2y + 2z + 5 = 0$ …..(1)

Given line is $\frac{x+3}{3} = \frac{y-2}{6} = \frac{z}{2}$ …..(2)

Equation of line through P (2, 3, 4) and parallel to the line (2) is

$\frac{x-2}{3} = \frac{y-3}{6} = \frac{z-4}{2}$ …..(3)

Any point on it is $Q\ (3k+2, 6k+3, 2k+4)$

If it lie on (1), then

$3(3k+2) + 2(6k+3) + 2\,(2k+4) + 5 = 0$

$\implies$ $25\,k + 25 = 0 \quad \implies k = -1$

$\therefore$ The required distance

$PQ = \sqrt{(2+1)^2 + (3+3)^2 + (4-2)^2} = \sqrt{49} = 7.$

55. Find the equation of the plane passing through the intersection of planes $2x + 3y - z = -1$ and $x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z = 4$. Also find the inclination of this plane with xy-plane. ***[Ans. $+13y + 4z = 90, cos^{-1}\left(\frac{4}{\sqrt{234}}\right)$]***

56. A man is kown to speak the truth 3 out of 4 times. He throws a die and reports that it is six. Find the probability that it is actually a six.

Solution:-

Let A = the event that man reports occurrence of 6

Let A_1 = the event of occurrence of 6 when a die is thrown and

A_2 = the event of non- occurrence of 6 when a die is thrown.

Now by Bayes' theorem,

$P\,(A_1/A) = \frac{P\,(A_1).P\,(A/A_1)}{P(A_1).P(A/A_1) + P(A_2).P\,(A/A_2)}$ …..(1)

$P\,(A_1) = \frac{1}{6}\,and\ (A_2) = \frac{5}{6}$

$P\ (A/A_1)$ = Probability that man reports occurrence of 6 when 6 has actually occurred.

= Probability that the man speaks the truth $= \frac{3}{4}$

$P\ (A/A_2) =$ Probability that man reports occurrence of 6 when 6 has not actually occurred.

= Probability that the man tells a lie $= \frac{1}{4}$

Putting these values in (1), we get

$$P\ (A_1/A)\ = \frac{\frac{1}{6}\cdot\frac{3}{4}}{\frac{1}{6}\cdot\frac{3}{4}+\frac{5}{6}\cdot\frac{1}{4}} = \frac{3}{8}$$

57. A box contains 2 gold and 3 silver coins. Another box contains 3 gold and 3 silver coins. A box is chosen random and a coin is drawn from it. If the selected coin is a gold coin, find the probability that it was drawn from the second box.

Solution:-

Let E_1 and E_2 be the events of selecting box I and box II respectively.

Let E be the event of selecting a gold coin. Then

$P\ (E_1) = \frac{1}{2} = P(E_2)$

$P\ (E/E_1) = \frac{2}{5}\ ; P\ (E/E_2)\ = \frac{3}{6} = \frac{1}{2}$

$\therefore$ By Bayes' theorem, the **required probability**

$$= P\ (E_2/E)\ = \frac{P\ (E_2).\ \ P\ (E/E_2)}{P\ (E_1)\ .\ P\ (E/E_1) + P\ (E_2).P\ (E/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{9}{20}}$$

$$= \frac{1}{4}\times\frac{20}{9} = \frac{5}{9}$$

58. A farmer has a supply of chemical fertilizer of type I which contains 10% nitrogen and 6% phosphoric acid and type II fertilizer which contains 5% nitrogen and 10% phosphoric acid. After testing the soil conditions of a field. It is found that atleast 14 kg of nitrogen and 14 kg of phosphoric acid is required for a good crop. The fertilizer type I costs Rs.2.00 per kg and type II costs Rs.3.00 per kg. how many kilograms of each fertilizer should be used to meet the requirement and the cost be minimum. ***[Ans. Minimum at (10.80) and is equal to Rs. 440]***

59. If a young man rides his motorcycle at 25 km/hr. he had to spend Rs. 2 per km on petrol. If he rides at a faster speed of 40 km/hr. the petrol cost increases at Rs. 5 per km. he has Rs. 100 to spend on petrol and wishes to find what is the maximum distance he can travel within one hour. Express this as LPP and solve it graphically. ***[Maximum at $\left(\frac{50}{3}.\frac{40}{3}\right)$]***

60. Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ are coplanar. Also find the equation of the plane containing them , ***[Ans. $x + y + z = 0$]***

61. Find the equation of the plane passing through, the point $(-1, 3, 2)$ and perpendicular to each of the places $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

[Ans. $7x - 8y + 3z + 25 = 0$]

62. A school wants to awards its students for the values of Honesty, Regularity and Hard work with a total cash awards of Rs. 6000. Three times the award money for hard work added to that given for Honesty amounts to Rs. 11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and hard work, suggest one more value which the school must include for awards.

Ans. Let awards money for Honesty, Regularity and Hard work be Rs. x, RS. y and Rs. z respectively.

According to given conditions,

$x + y + z = 6000$

$\Rightarrow$ $x + y + z = 6000$

x + 3z = 11000

$\Rightarrow \quad x + 0y + 3z = 11000$

$x + z = 2y$

$\Rightarrow \quad x - 2y + z = 0$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

AX = B is matrix equation, its equation, its solution

Is $X = A^{-1}B,$ …(i)

Where $A^{-1} = \frac{1}{|A|}\ adj\ A$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$= 1(6) - 1(-2) + 1(-2)$

$= 6 \neq 0.$

Cofactors of each element in $|A|$ are

$A_{11} = (0 + 6) = 6$	$A_{12} = -(1 - 3) = 2$
$A_{13} = (-2 - 0) = -2$	$A_{21} = -(1 + 2) = -3$
$A_{22} = (1 - 1) = 0$	$A_{23} = -(-2 - 1) = 3$
$A_{31} = (3 - 0) = 3$	$A_{32} = -(3 - 1) = -2$
$A_{33} = (0 - 1) = -1$	

$$\text{Adj A} = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}'$$

$$= \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$X = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

[From (i)]

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 - 0 \\ -12000 + 33000 - 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$\Rightarrow \quad x = 500, y = 2000, z = 3500$

$\therefore$ Award money for Honesty,

x = Rs.500

Award money for Regularity,

y = Rs. 2000

Award money for Hard work,

z = Rs. 3500

The value which should be included is Empathy.

63. 10 students were selected from a school on the basis of values for giving awards and were divided into three groups. The first group comprises hard workers, the second group has honest and law abiding students and the third group contains Vigilant and Obedient students. Double the number of students of the first group added to the number in the second group gives 13, while the combined strength of first and second group is four times that of the third group. Using matrix method, find the number of students in each group. Apart from the values, Hard

work, Honesty and Respect for law, Vigilance and Obedience, suggest one more value, which in your opinion, the school should consider for awards.

Ans. Let there be x students in Hard workers, y students in honest and law abiding students and z students in Vigilant and Obedient group.

According to given conditions

$x + y + z = 10$

$2x + y = 13$

$x + y - 4z = 0$

Corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

i.e. AX = B

Its solution is X = $A^{-1} B$, ...(i)

Where $A^{-1} = \frac{1}{|A|} adj\ A$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{vmatrix}$$

$= 1(-4) - 1(-8) + 1(1)$

$= -4 + 8 + 1 = 5 \neq 0$

Cofactors of each element in $|A|$ are

$A_{11} = (-4 - 0) = -4$ $A_{12} = -(-8 - 0) = 8$

$A_{13} = (2 - 1) = 1$ $A_{21} = -(-4 - 1) = 5$

$A_{22} = (-4 - 1) = -5$ $A_{23} = -(1 - 1) = 0$

$A_{31} = (0 - 1) = -1$ $A_{32} = -(0 - 2) = 2$

$A_{33} = (1 - 2) = -1$

$$\text{Adj A} = \begin{bmatrix} -4 & 8 & 1 \\ 5 & -5 & 0 \\ -1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{5} \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

From (i), we get

$$X = \frac{1}{5} \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -40 + 65 - 0 \\ 80 - 65 + 0 \\ 10 + 0 - 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 25 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad x = 5, y = 3, z = 2$$